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A Method of Integrating the Square Roots of Quadratics.

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A METHOD OF INTEGRATING THE SQUARE ROOTS OF QUADRATICS.

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CONTENTS.

PART I. Upon $\int \sin^p\theta \cos^q\theta$ . . . . . (*a*)

PART II. Upon $\int (a + bx + cx^2)^{\frac{n}{2}} x^m dx$ . . . . . (*b*)

PART I.

To integrate the general form, $\int \sin^p\theta \cos^q\theta d\theta$ . . . . . (*a*), in all cases where p and q are integers, positive or negative.

Apply to (*a*) the formula for integrating by parts, viz.: $\int udv = vu - \int vdu$, by making $u = \sin^{p-1}\theta \cos^{q-1}\theta$ and $dv = \sin\theta \cos\theta$; $\therefore v = \frac{\sin^2\theta}{2}$, or $v = -\frac{\cos^2\theta}{2}$. If $v = \frac{\sin^2\theta}{2}$, we obtain formula (1) of the succeeding summary; but if $v = -\frac{\cos^2\theta}{2}$, we then obtain formula (2).

Again, since $1 = \sin^2\theta + \cos^2\theta$, $\int \sin^p\theta \cos^q\theta d\theta$,

$$= \int \sin^{p+2}\theta \cos^q\theta d\theta + \int \sin^p\theta \cos^{q+2}\theta d\theta \quad . \quad . \quad . \quad . \quad (c)$$

$$= \int \sin^p\theta \cos^{q-2}\theta d\theta - \int \sin^{p+2}\theta \cos^{q-2}\theta d\theta \quad . \quad . \quad . \quad (d)$$

$$= \int \sin^{p-2}\theta \cos^q\theta d\theta - \int \sin^{p-2}\theta \cos^{q+2}\theta d\theta \quad . \quad . \quad . \quad (e)$$

1st. Substitute in place of the last term of (*c*) its value as given by (1), from which process we shall obtain formula (3) of the summary.

2d. Substitute in place of the last term but one of (*c*), its value as given by (2), from which we shall obtain formula (4).

3d. Substitute in place of the last term of (*d*) its value as given by (2), from which we shall obtain formula (5).

4th. Substitute in place of the last term of (*e*) its value as given by (1), from which we shall obtain formula (6).

SUMMARY.

$$\int \sin^p\theta \cos^q\theta d\theta \quad \ldots \ldots \ldots

or upon

$$\int \frac{d\theta}{\cos\theta} = \int \frac{\cos\theta d\theta}{\cos^2\theta} = \frac{1}{2}\left[\int \frac{\cos\theta d\theta}{1+\sin\theta} + \int \frac{\cos\theta d\theta}{1-\sin\theta}\right] = \frac{1}{2}\log_e\left[\frac{1+\sin\theta}{1-\sin\theta}\right]$$

$$= \log_e[\sec\theta + \tang\theta] = \log_e \tang\left(\frac{\theta}{2} + \frac{\pi}{4}\right).$$

5. If p and q are both negative and even, by (3) and (4), (a) will finally depend either upon

$$\int \frac{d\theta}{\sin^2\theta} = -\cotang\theta, \quad \text{or upon} \quad \int \frac{d\theta}{\cos^2\theta} = \tang\theta.$$

6. If p and q are both negative and odd, by (3) and (4), (a) will depend upon

$$\int \frac{d\theta}{\sin\theta\cos\theta} = \int \frac{(\sin^2\theta + \cos^2\theta)}{\sin\theta\cos\theta} d\theta = \int \frac{\sin\theta}{\cos\theta} d\theta + \int \frac{\cos\theta}{\sin\theta} d\theta$$

$$= \log_e \sin\theta - \log_e \cos\theta = \log_e \tang\theta = -\log_e \cotang\theta.$$

7. If p and q are one odd and the other even, one positive and the other negative.

When the odd number is numerically the larger, by (1) and (2), (a) will depend upon one of the four following forms (n being an integer):

$$\int \sin^{2n+1}\theta d\theta, \quad \int \cos^{2n+1}\theta d\theta, \quad \int \frac{d\theta}{\sin^{2n+1}\theta}, \quad \int \frac{d\theta}{\cos^{2n+1}\theta};$$

which forms have been already integrated.

When the even number is numerically the larger, we have similarly, by (1) and (2), four forms:

$\int \frac{\sin^{2n}\theta d\theta}{\cos\theta}$, which, by (6), depends on $\int \frac{d\theta}{\cos\theta} = \log_e \tang\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$,

$\int \frac{\cos^{2n}\theta d\theta}{\sin\theta}$, " " (5), " " $\int \frac{d\theta}{\sin\theta} = \log_e \tang\frac{\theta}{2}$,

$\int \frac{\sin\theta d\theta}{\cos^{2n}\theta}$, " " (4), " " $\int \sin\theta d\theta = -\cos\theta$,

$\int \frac{\cos\theta d\theta}{\sin^{2n}\theta}$, " " (3), " " $\int \cos\theta d\theta = \sin\theta$.

8. If p and q are both even, one positive and the other negative, by (1) and (2) we have the four following forms:

$$\int \sin^{2n}\theta d\theta, \quad \int \frac{d\theta}{\sin^{2n}\theta} \quad \int \cos^{2n}\theta d\theta, \quad \int \frac{d\theta}{\cos^{2n}\theta};$$

which forms have already been integrated.

9. If p and q are both odd, one positive and the other negative, by (1) and (2) we have the four forms:

$\int \frac{\sin^{2n-1}\theta d\theta}{\cos\theta}$. which, by (6), depends on $\int \frac{\sin\theta d\theta}{\cos\theta} = -\log_e \cos\theta$.

$\int \frac{\cos^{2n-1}\theta d\theta}{\sin\theta}$ " " (5), " " $\int \frac{\cos\theta d\theta}{\sin\theta} = \log_e \sin\theta$.

$\int \frac{\sin\theta d\theta}{\cos^{2n+1}\theta}$, " " (4), " " $\int \frac{\sin\theta d\theta}{\cos^3\theta} = -\log_e \cos^2\theta$.

$\int \frac{\cos\theta d\theta}{\sin^{2n+1}\theta}$, " " (3), " " $\int \frac{\cos\theta d\theta}{\sin^3\theta} = \log_e \sin\theta$.

When, however, $(p+q)=0$, the above are failing cases of (5) and (6), but not of (1) and (2).

SUMMARY.

We can, therefore, by the use of formulas (1) to (6), inclusive, cause expression (a) to depend upon one of the following eight forms, provided that p and q are integers:

$$\int \sin\theta d\theta = -\cos\theta \quad \ldots \quad (7)$$

$$\int \cos\theta d\theta = \sin\theta \quad \ldots \quad (8)$$

$$\int \frac{\sin\theta d\theta}{\cos\theta} = -\log_e \cos\theta \quad \ldots \quad (9)$$

$$\int \frac{\cos\theta d\theta}{\sin\theta} = \log_e \sin\theta \quad \ldots \quad (10)$$

$$\int \frac{d\theta}{\sin\theta} = \log_e \left[\frac{1-\cos\theta}{\sin\theta}\right] = \log_e \operatorname{tang}\frac{\theta}{2} \quad \ldots \quad (11)$$

$$\int \frac{d\theta}{\cos\theta} = \log_e \left[\frac{1+\sin\theta}{\cos\theta}\right] = \log_e \operatorname{tang}\left[\frac{\theta}{2}+\frac{\pi}{4}\right] \quad \ldots \quad (12)$$

$$\int \frac{d\theta}{\sin\theta\cos\theta} = \log_e \operatorname{tang}\theta = -\log_e \operatorname{cotang}\theta \quad \ldots \quad (13)$$

$$\int d\theta = \theta \quad \ldots \quad (14)$$

EXAMPLES.

1. $\int \frac{d\theta}{\sin^2\theta} = -\operatorname{cotang}\theta$.

2. $\int \frac{d\theta}{\cos^2\theta} = \operatorname{tang}\theta$.

3. $\int \frac{\sin\theta d\theta}{\cos^2\theta} = \sec\theta = \frac{1}{\cos\theta}.$

4. $\int \frac{\cos\theta d\theta}{\sin^2\theta} = -\operatorname{cosec}\theta = -\frac{1}{\sin\theta}.$

5. $\int \sin^2\theta d\theta = \frac{1}{2}(\theta - \sin\theta\cos\theta).$

6. $\int \cos^2\theta d\theta = \frac{1}{2}(\theta + \sin\theta\cos\theta).$

7. $\int \frac{\sin^2\theta d\theta}{\cos^2\theta} = \operatorname{tang}\theta - \theta.$

8. $\int \frac{\cos^2\theta d\theta}{\sin^2\theta} = -\operatorname{cotang}\theta - \theta.$

9. $\int \frac{d\theta}{\sin^2\theta\cos^2\theta} = \operatorname{tang}\theta - \operatorname{cotang}\theta.$

10. $\int \sin\theta\cos\theta d\theta = \frac{1}{2}\sin^2\theta,$ or $= -\frac{1}{2}\cos^2\theta.$

It is possible to apply each of the formulæ from (1) to (6), inclusive, to that part of itself which is still under the sign of integration, or to the corresponding part of either of the others. This operation can be repeated at will, and we thus obtain formulæ which are some of them useful, as they give at once the integral of expressions which it would otherwise require several processes to effect.

By repeatedly applying each of the formulæ (1) to (6) to *itself*, as suggested, we have the general forms of such expansions in the six following formulæ, which we arrive at after effecting n integrations (n being an integer greater than unity). These formulæ may be established by mathematical induction.

$$\int \sin^p\theta \cos^q\theta d\theta,$$

by (1):

$$= \frac{1}{p+1}\sin^{p+1}\theta\cos^{q-1}\theta + \frac{(q-1)}{(p+1)}\cdot\frac{1}{(p+3)}\sin^{p+3}\theta\cos^{q-3}\theta +$$

$$\frac{(q-1)}{(p+1)}\cdot\frac{(q-3)}{(p+3)}\cdot\frac{1}{(p+5)}\sin^{p+5}\theta\cos^{q-5}\theta + \quad \ldots\ldots\ldots \quad +$$

$$\frac{(q-1)(q-3)\ldots\ldots[q+1-2(n-1)]}{(p+1)(p+3)\ldots\ldots[p-1+2(n-1)]}\cdot\frac{\sin^{p-1+2n}\theta\cos^{q+1-2n}\theta}{[p+1+2(n-1)]} \quad +$$

$$\frac{(q-1)(q-3)\ldots\ldots(q+1-2n)}{(p+1)(p+3)\ldots\ldots(p-1+2n)}\int \sin^{p+2n}\theta\cos^{q-2n}\theta d\theta \quad \ldots \quad (15)$$

by (2):

$$= -\frac{1}{q+1}\sin^{p-1}\theta\cos^{q+1}\theta - \frac{(p-1)}{(q+1)}\cdot\frac{1}{(q+3)}\sin^{p-3}\theta\cos^{q+3}\theta -$$

$$\frac{(p-1)}{(q+1)}\cdot\frac{(p-3)}{(q+3)}\cdot\frac{1}{(q+5)}\sin^{p-5}\theta\cos^{q+5}\theta - \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad -$$

$$\frac{(p-1)(p-3)\ldots\ldots[p+1-2(n-1)]}{(q+1)(q+3)\ldots\ldots[q-1+2(n-1)]}\cdot\frac{\sin^{p+1-2n}\theta\cos^{q-1+2n}\theta}{[q+1+2(n-1)]} -$$

$$\frac{(p-1)(p-3)\ldots\ldots(p+1-2n)}{(q+1)(q+3)\ldots\ldots(q-1+2n)}\int\sin^{p-2n}\theta\cos^{q+2n}\theta d\theta \quad . \quad . \quad (16)$$

by (3):

$$= \frac{1}{p+1}\sin^{p+1}\theta\cos^{q+1}\theta + \frac{(p+q+2)}{(p+1)}\cdot\frac{1}{(p+3)}\sin^{p+3}\theta\cos^{q+1}\theta +$$

$$\frac{(p+q+2)}{(p+1)}\cdot\frac{(p+q+4)}{(p+3)}\cdot\frac{1}{(p+5)}\sin^{p+5}\theta\cos^{q+1}\theta + \quad . \quad . \quad . \quad . \quad +$$

$$\frac{(p+q+2)(p+q+4)\ldots\ldots[p+q+2(n-1)]}{(p+1)\quad(p+3)\ldots\ldots[p-1+2(n-1)]}\cdot\frac{\sin^{p-1+2n}\theta\cos^{q+1}\theta}{[p+1+2(n-1)]} +$$

$$\frac{(p+q+2)(p+q+4)\ldots\ldots(p+q+2n)}{(p+1)(p+3)\ldots\ldots[p+1+2(n-1)]}\int\sin^{p+2n}\theta\cos^{q}\theta d\theta \quad . \quad . \quad (17)$$

by (4):

$$= -\frac{1}{q+1}\sin^{p+1}\theta\cos^{q+1}\theta - \frac{(p+q+2)}{(q+1)}\cdot\frac{1}{(q+3)}\sin^{p+1}\theta\cos^{q+3} -$$

$$\frac{(p+q+2)}{(q+1)}\cdot\frac{(p+q+4)}{(q+3)}\cdot\frac{1}{(q+5)}\sin^{p-1}\theta\cos^{q+5}\theta - \quad . \quad . \quad . \quad . \quad . \quad -$$

$$\frac{(p+q+2)(p+q+4)\ldots\ldots[p+q+2(n-1)]}{(q-1)(q+3)\ldots\ldots[q-1+2(n-1)]}\cdot\frac{\sin^{p+1}\theta\cos^{q-1+2n}\theta}{[q+1+2(n-1)]} +$$

$$\frac{(p+q+2)(p+q+4)\ldots\ldots(p+q+2n)}{(q+1)(q+3)\ldots\ldots[q+1+2(n-1)]}\int\sin^{p}\theta\cos^{q-2n}\theta d\theta \quad . \quad . \quad (18)$$

by (5):

$$= \frac{1}{p+q}\sin^{p+1}\theta\cos^{q-1}\theta + \frac{(q-1)}{(p+q)}\cdot\frac{1}{(p+q-2)}\sin^{p+1}\theta\cos^{q-3}\theta +$$

$$\frac{(q-1)}{(p+q)}\cdot\frac{(q-3)}{(p+q-2)}\cdot\frac{1}{(p+q-4)}\sin^{p+1}\theta\cos^{q-5}\theta + \quad . \quad . \quad . \quad . \quad . \quad +$$

$$\frac{(q-1)(q-3)\ldots\ldots[q+1-2(n-1)]}{(p+q)(p+q-2)\ldots\ldots[p+q+2-2(n-1)]}\cdot\frac{\sin^{p+1}\theta\cos^{q+1-2n}\theta}{[p+q-2(n-1)]} +$$

$$\frac{(q-1)(q-3)\ldots\ldots(q+1-2n)}{(p+q)(p+q-2)\ldots\ldots[p+q-2(n-1)]}\int\sin^{p}\theta\cos^{q-2n}\theta d\theta \quad (19)$$

by (6):

$$= -\frac{1}{p+q}\sin^{p-1}\theta\cos^{q+1}\theta - \frac{(p-1)}{(p+q)}\cdot\frac{1}{(p+q-2)}\sin^{p-3}\theta\cos^{q+1}\theta -$$

$$\frac{(p-1)}{(p+q)}\cdot\frac{(p-3)}{(p+q-2)}\cdot\frac{1}{(p+q-4)}\sin^{p-5}\theta\cos^{q+1}\theta - \quad . \quad . \quad . \quad . \quad . \quad -$$

$$\frac{(p-1)(p-3)\ldots.[p+1-2(n-1)]}{(p+q)(p+q-2)\ldots.[p+q+2-2(n-1)]}\cdot\frac{\sin^{p+1-2n}\delta\cos^{q+1}\theta}{[p+q-2(n-1)]} +$$

$$\frac{(p-1)(p-3)\ldots\ldots[p+1-2n]}{(p+q)(p+q-2)\ldots\ldots[p+q-2(n-1)]}\int\sin^{p-2n}\theta\cos^{q}\theta d\theta \quad (20)$$

If formulæ (1) to (6), inclusive, may be called *primaries*, we shall now give six *secondary* formulæ, (21) to (26), inclusive, which can be obtained by one operation of one of the primaries upon one of the others. These six (viz.: (21) to (26)), together with the six contained in the general formulæ (15) to (20), inclusive, are all the secondaries obtainable, and they may serve as an example of the tertiaries, &c., possible:

$$\int\sin^{p}\theta\cos^{q}\theta d\theta,$$

by applying (1) to (3):

$$= \frac{1}{p+1}\sin^{p+1}\theta\cos^{q-1}\theta + \frac{(q-1)}{(p+1)(p+3)}\sin^{p+3}\theta\cos^{q-1}\theta +$$

$$\frac{(q-1)(p+q+2)}{(p+1)(p+3)}\int\sin^{p+4}\theta\cos^{q-2}\theta d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

by applying (2) to (4),

$$= -\frac{1}{q+1}\sin^{p-1}\theta\cos^{p+1}\theta - \frac{(p-1)}{(q+1)(q+3)}\sin^{p-1}\theta\cos^{q+3}\theta +$$

$$\frac{(p-1)(p+q+2)}{(q+1)(q+3)}\int\sin^{p-2}\theta\cos^{q+4}\theta d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

by applying (1) to (5):

$$= \frac{1}{p+q}\sin^{p+1}\theta\cos^{q-1}\theta + \frac{(q-1)}{(p+q)(p+1)}\sin^{p+1}\theta\cos^{q-3}\theta +$$

$$\frac{(q-1)(q-3)}{(p+q)(p+1)}\int\sin^{p+2}\theta\cos^{q-4}\theta d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

by applying (2) to (6):

$$= -\frac{1}{p+q}\sin^{p-1}\theta\cos^{q+1}\theta - \frac{(p-1)}{(p+q)(q+1)}\sin^{p-3}\theta\cos^{q+1}\theta +$$

$$\frac{(p-1)(p-3)}{(p+q)(q+1)}\int\sin^{p-4}\theta\cos^{q+2}\theta d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

by applying (3) to (4):

$$= \frac{1}{p+1} \sin^{p+1}\theta \cos^{q+3}\theta - \frac{1}{q+1} \sin^{p+3}\theta \cos^{q+1}\theta +$$

$$\frac{(p+q+2)(p+q+4)}{(p+1)(q+1)} \int \sin^{p+2}\theta \cos^{q+2}\theta d\theta \quad \ldots\ldots\ldots \quad (25)$$

by applying (5) to (6):

$$= \frac{(p-1)}{(p+q)(p+q-2)} \sin^{p+1}\theta \cos^{q-1}\theta - \frac{(q-1)}{(p+q)(p+q-2)} \sin^{p-1}\theta \cos^{q+1}\theta +$$

$$\frac{(p-1)(q-1)}{(p+q)(p+q-2)} \int \sin^{p-2}\theta \cos^{q-2}\theta d\theta \quad \ldots\ldots\ldots \quad (26)$$

PART II.

To integrate the general form, $u = \int (a + bx + cx^2)^{\frac{n}{2}} x^m dx$ . . (*b*)

in all cases when m and n are integers, positive or negative, provided the constants a, b, and c are such as to render the integral real.

By change of form, we have, if

$$u = \int (a + bx + cx^2)^{\frac{n}{2}} x^m dx \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (b)$$

$$u = \sqrt{-c^n} \int \left[\frac{b^2 - 4ac}{4c^2} - \left(x + \frac{b}{2c}\right)^2\right]^{\frac{n}{2}} x^m dx \quad . \; . \; . \; (f)$$

$$u = \sqrt{-a^n} \int \left[\frac{b^2 - 4ac}{4a^2} - \left(x^{-1} + \frac{b}{2a}\right)^2\right]^{\frac{n}{2}} x^{m+n} dx \quad . \; . \; (g)$$

$$u = \sqrt{c^n} \int \left[\left(x + \frac{b}{2c}\right)^2 - \frac{b^2 - 4ac}{4c^2}\right]^{\frac{n}{2}} x^m dx \; . \; . \; . \; . \; . \; (h)$$

$$u = \sqrt{a^n} \int \left[\left(x^{-1} + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]^{\frac{n}{2}} x^{m+n} dx \quad . \; . \; . \; (k)$$

We shall now point out briefly what transformations will cause these forms to depend for their integration upon that of (*a*).

1st. Formula (*f*) may be integrated when m is positive and $r^2 = \frac{b^2 - 4ac}{4c^2} > 0$; for, let $x + \frac{b}{2c} = r \sin\theta$; $\therefore \; dx = r\cos\theta d\theta$, and $\left[\frac{b^2 - 4ac}{4c^2} - \left(x + \frac{b^2}{2c}\right)^2\right]^{\frac{n}{2}} = r^n \cos^n\theta$. Substituting, we have,

$$u = r^{n+m+1} \sqrt{-c^n} \int \left[\sin\theta - \frac{b}{2cr}\right]^m \cos^{n+1}\theta d\theta \; . \; . \; . \; (27)$$

which is real when c is negative, and is integrated by expanding, multiplying, and thus obtaining several terms of the form of (*a*).

This transformation might have been effected with equal ease by putting $x + \frac{b}{2c} = r\cos\theta$, etc., etc.

2d. Formula (g) may be integrated when m is negative, and $r^2 = \frac{b^2 - 4ac}{4a^2} > 0$; for, let $x^{-1} + \frac{b}{2a} = r\sin\theta$; $\therefore\ -\frac{dx}{x^2} = r\cos\theta d\theta$, and $\left[\frac{b^2-4ac}{4a^2} - \left(x^{-1} + \frac{b}{2a}\right)^2\right]^{\frac{n}{2}} = r^n\cos^n\theta$. Substituting, we have,

$$u = -r^{-m-1}\sqrt{-a^n}\int\left[\sin\theta - \frac{b}{2ar}\right]^{-(m+n+2)}\cos^{n+1}\theta d\theta \quad . . \quad (28)$$

which is real when a is negative, and it is integrable whenever we can make $-(m+n+2) > 0$, in which m is negative. This we can always effect by making the exponent of the quadratic any minus number desirable, as follows:

Multiply and divide (g) by

$$\left[\frac{b^2-4ac}{4a^2} - \left(x^{-1} + \frac{b}{2c}\right)^2\right]^{\frac{s}{2}},$$

in which $-(m-s+2) > 0$, and expand the numerator to the power indicated by the exponent $\frac{s+n}{2}$ (which exponent must be an integer when s and n are odd). Thus we obtain several terms, each of which can be integrated by (28).

3d. Formula (h) may be integrated when m is positive and $r^2 = \frac{b^2-4ac}{4c^2} > 0$; for, let $x + \frac{b}{2c} = r\sec\theta$; $\therefore\ dx = \frac{r\sin\theta d\theta}{\cos^2\theta}$, and $\left[\left(x + \frac{b}{2c}\right)^2 - \frac{b^2-4ac}{4c^2}\right]^{\frac{n}{2}} = r^n\tang^n\theta$. Substituting, we have,

$$u = r^{n+m+1}\sqrt{c^n}\int\left[\sec\theta - \frac{b}{2cr}\right]^m \frac{\tang^{n+1}\theta d\theta}{\cos\theta} \quad . \quad . \quad . \quad (29)$$

which is real when c is positive, and depends upon (a) in the same manner as (27) and (28).

4th. Formula (k) may be integrated when m is negative, and $r^2 = \frac{b^2-4ac}{4a^2} > 0$: for, let $\left(x^{-1} + \frac{b}{2a}\right) = r\sec\theta$; $\therefore\ -\frac{dx}{x^2} = \frac{r\sin\theta d\theta}{\cos^2\theta}$ and $\left[\left(x^{-1} + \frac{b}{2c}\right)^2 - \frac{b^2-4ac}{4a^2}\right]^{\frac{n}{2}} = r^n\tang^n\theta$. Substituting, we have,

$$u = -r^{-m-1}\sqrt{a^n}\int\left[\sec\theta - \frac{b}{2ar}\right]^{-(m+n+2)} \frac{\tang^{n+1}\theta d\theta}{\cos\theta} \quad . \quad . \quad (30)$$

which is real when a is positive, and it is integrable when we render (as previously) $-(m+n+2) > 0$.

5th. Formula (h) may be integrated when m is positive and $r^2 = -\frac{b^2-4ac}{4c^2} > 0$; for, let $x + \frac{b}{2c} = r\,\text{tang}\theta$; $\therefore\ dx = \frac{rd\theta}{\cos^2\theta}$. Substitute, and we have,

$$u = r^{n+m+1}\sqrt{c^n}\int\left[\text{tang}\theta - \frac{b}{2cr}\right)^m \frac{\sec^n\theta d\theta}{\cos^2\theta} \quad . \ . \ . \quad (31)$$

6th. Formula (k) may be integrated when m is negative and $r^2 = -\frac{b^2-4ac}{4a^2} > 0$; for let $x^{-1} + \frac{b}{4a} = r\,\text{tang}\theta$; $\therefore\ -\frac{dx}{x^2} = \frac{rd\theta}{\cos^2\theta}$. Substitute, and we have,

$$u = -r^{-m-1}\sqrt{a^n}\int\text{tang}\theta - \frac{b}{2ar}\Big]^{-(m+n+2)} \frac{\sec^n\theta d\theta}{\cos^2\theta} \quad . \ . \quad (32)$$

SUMMARY.

$$u = \int(a + bx + cx^2)^{\frac{n}{2}}x^m dx \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (b)$$

If $x + \frac{b}{2c} = r\sin\theta$,

$$u = r^{m+n+1}\sqrt{-c^n}\int\left[\sin\theta - \frac{b}{2cr}\right]^m \cos^{n+1}\theta d\theta \ . \ . \ . \ . (27)$$

If $x^{-1} + \frac{b}{2a} = r\sin\theta$,

$$u = -\frac{\sqrt{-a^n}}{r^{m+1}}\int\left[\sin\theta - \frac{b}{2ar}\right]^{-(m+n+2)} \cos^{n+1}\theta d\theta \quad . \ . \ (28)$$

If $x + \frac{b}{2c} = r\sec\theta$,

$$u = r^{m+n+1}\sqrt{c^n}\int\left[\frac{1}{\cos\theta} - \frac{b}{2cr}\right]^m \frac{\sin^{n+1}\theta d\theta}{\cos^{n+2}\theta} \ . \ . \ . \ . (29)$$

If $x^{-1} + \frac{b}{2a} = r\sec\theta$,

$$u = -\frac{\sqrt{a^n}}{r^{m+1}}\int\left[\frac{1}{\cos\theta} - \frac{b}{2ar}\right]^{-(m+n+2)} \frac{\sin^{n+1}\theta d\theta}{\cos^{n+2}\theta} \ . \ . \ . \ . (30)$$

If $x + \frac{b}{2c} = \text{tang}\theta$,

$$u = r^{m+n+1}\sqrt{c^n}\int\left[\frac{\sin\theta}{\cos\theta} - \frac{b}{2cr}\right]^m \frac{d\theta}{\cos^{n+2}\theta} \ . \ . \ . \ . \ . (31)$$

If $x^{-1} + \frac{b}{2a} = r\,\text{tang}\theta$,

$$u = -\frac{\sqrt{a^n}}{r^{m+1}}\int\left[\frac{\sin\theta}{\cos\theta} - \frac{b}{2ar}\right]^{-(m+n+2)} \frac{d\theta}{\cos^{n+2}\theta} \ . \ . \ . \ . \ . (32)$$

EXAMPLES.

1. $\int (a + bx + cx^2)^{\frac{p}{2}}(h + kx)^q dx.$

In this let $h + kx = x^1$; then the integral will be of the form (b), if p and q are integers.

2. $\int (a + bx)^{\frac{p}{2}}(h + kx)^{\frac{q}{2}}x^m dx$

$= \int [(a + bx)(h + kx)]^{\frac{p}{2}}(h + kx)^{\frac{q-p}{2}}x^m dx$, when $\frac{q-p}{2} > 0$;

or,

$= \int [(a + bx)(h + kx)^{\frac{q}{2}}(a + bx)^{\frac{p-q}{2}}x^m dx$, when $\frac{p-q}{2} > 0$.

Expand the binomial to the power $\pm \frac{p-q}{2}$. Since that is a positive integer, multiply and integrate the several terms each of the form (b).

3. $\int (a + cx^2)^{\frac{p}{2}}(h + ex^2)^{\frac{q}{2}}x^m dx$ may, in many cases, be transformed into (a) either by making $(a + cx^2) = r^2 \tan g^2\theta$ and $r^2 = -\frac{1}{h}(ah - ce)$ or $(ax^{-2} + c) = r^2 \tan g^2\theta$ and $r^2 = \frac{1}{e}(ah - ce)$. A full discussion of this form might be made, similar in nature to the discussion of form (b).

4. $\int \frac{(1+cx^2)x^{p+q-2}dx}{(1-cx^2)^q(1+ax^2+c^2x^4)^{\frac{p}{2}}} = \int \frac{(x^{-2}+c)dx}{(x^{-1}-cx)^q[(x^{-1}-cx)^2+a+2c]^{\frac{p}{2}}},$

In this let $(x^{-1} + cx) = r \tan g\theta$ and $r^2 = a + 2c$.

5. $\int \frac{(b+ex^2+bc^2x^4)x^{p-1}dx}{(1-c^2x^4)(1+ax^2+c^2x^4)^{\frac{p}{2}}}$

$= \frac{2bc+e}{4c}\int \frac{(1+cx^2)x^{p-1}dx}{(1-cx^2)(1+ax^2+c^2x^4)^{\frac{p}{2}}} + \frac{2bc-e}{4c}\int \frac{(1-cx^2)x^{p-1}dx}{(1+cx^2)(1+ax^2+c^2x^4)^{\frac{p}{2}}},$

which is a case of Example 4.

It may be stated, in conclusion, that the method herein briefly sketched, by which integrals containing some power of the square root of a quadratic are transformed and made to depend on form (a), has been found by the author, in practice, to be practicable, expeditious, and useful, especially when a proper transformation of limits is effected at the same time as the first-mentioned transformation.

NOTE A.

The following tables of relations between circular functions will facilitate the necessary transformations.

TABLE I.

Relations between the Direct Circular Functions.

$\sin\theta$	$=\sqrt{1-\cos^2\theta}$	$=\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$=\frac{1}{\sqrt{1+\cot^2\theta}}$	$=\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$=\frac{1}{\operatorname{cosec}\theta}$
$\sqrt{1-\sin^2\theta}$	$=\cos\theta$	$=\frac{1}{\sqrt{1+\tan^2\theta}}$	$=\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$=\frac{1}{\sec\theta}$	$=\frac{\sqrt{\operatorname{cosec}^2\theta-1}}{\operatorname{cosec}\theta}$
$\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$=\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$=\tan\theta$	$=\frac{1}{\cot\theta}$	$=\sqrt{\sec^2\theta-1}$	$=\frac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$=\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$=\frac{1}{\tan\theta}$	$=\cot\theta$	$=\frac{1}{\sqrt{\sec^2\theta-1}}$	$=\sqrt{\operatorname{cosec}^2\theta-1}$
$\frac{1}{\sqrt{1-\sin^2\theta}}$	$=\frac{1}{\cos\theta}$	$=\sqrt{1+\tan^2\theta}$	$=\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$=\sec\theta$	$=\frac{\operatorname{cosec}\theta}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\frac{1}{\sin\theta}$	$=\frac{1}{\sqrt{1-\cos^2\theta}}$	$=\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$=\sqrt{1+\cot^2\theta}$	$=\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$=\operatorname{cosec}\theta$

TABLE II.

Relations between the Inverse Circular Functions.

$\sin^{-1}x$	$=\cos^{-1}\sqrt{1-x^2}$	$=\tan^{-1}\frac{x}{\sqrt{1-x^2}}$	$=\cot^{-1}\frac{\sqrt{1-x^2}}{x}$	$=\sec^{-1}\frac{1}{\sqrt{1-x^2}}$	$=\operatorname{cosec}^{-1}\frac{1}{x}$
$\sin^{-1}\sqrt{1-x^2}$	$=\cos^{-1}x$	$=\tan^{-1}\frac{\sqrt{1-x^2}}{x}$	$=\cot^{-1}\frac{x}{\sqrt{1-x^2}}$	$=\sec^{-1}\frac{1}{x}$	$=\operatorname{cosec}^{-1}\frac{1}{\sqrt{1-x^2}}$
$\sin^{-1}\frac{x}{\sqrt{1+x^2}}$	$=\cos^{-1}\frac{1}{\sqrt{1+x^2}}$	$=\tan^{-1}x$	$=\cot^{-1}\frac{1}{x}$	$=\sec^{-1}\sqrt{+1x^2}$	$=\operatorname{cosec}^{-1}\frac{\sqrt{1+x^2}}{x}$
$\sin^{-1}\frac{1}{\sqrt{1+x^2}}$	$=\cos^{-1}\frac{x}{\sqrt{1+x^2}}$	$=\tan^{-1}\frac{1}{x}$	$=\cot^{-1}x$	$=\sec^{-1}\frac{\sqrt{1+x^2}}{x}$	$=\operatorname{cosec}^{-1}\sqrt{1+x^2}$
$\sin^{-1}\frac{\sqrt{x^2-1}}{x}$	$=\cos^{-1}\frac{1}{x}$	$=\tan^{-1}\sqrt{x^2-1}$	$=\cot^{-1}\frac{1}{\sqrt{x^2-1}}$	$=\sec^{-1}x$	$=\operatorname{cosec}^{-1}\frac{x}{\sqrt{x^2-1}}$
$\sin^{-1}\frac{1}{x}$	$=\cos^{-1}\frac{\sqrt{x^2-1}}{x}$	$=\tan^{-1}\frac{1}{\sqrt{x^2-1}}$	$=\cot^{-1}\sqrt{x^2-1}$	$=\sec^{-1}\frac{x}{\sqrt{x^2-1}}$	$=\operatorname{cosec}^{-1}x$

NOTE B.

The symmetrical manner in which formulæ (1) to (6), inclusive, are obtained may be further shown by the following process of obtaining the ordinary reduction formulæ for the form,

$$\int X^p x^{m-1} dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b^1)$$

in which $X = a + bx^n$. Integrating by parts, the parts being indicated by the period,

$$\int X^p . \, x^{m-1} dx = \frac{X^p x^m}{m} - \frac{bnp}{m} \int X^{p-1} x^{m+n-1} dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or,

$$\int x^{m-n} . \, X x^{n-1} dx = \frac{X^{p+1} x^{m-n}}{bn(p+1)} - \frac{m-n}{bn(p+1)} \int X^{p+1} x^{m-n-1} dx \quad . \quad . \quad (2)$$

It is to be noticed that (1) and (2) are the *only* integrations by parts of (b^1) which are *binomials*. Again, we have identically by separation,

$$\int X^p x^{m-1} dx = a \int X^{p-1} x^{m-1} dx + b \int X^{p-1} x^{m+n-1} dx \quad . \quad . \quad (s^1)$$

$$\int X^{p+1} x^{m-1} dx = a \int X^p x^{m-1} dx + b \int X^p x^{m+n-1} dx \quad . \quad . \quad . \quad (s)$$

$$\int X^{p+1} x^{m-n-1} dx = a \int X^p x^{m-n-1} dx + b \int X^p x^{m-1} dx . \quad . \quad . \quad (s_1)$$

It is to be noticed that (s^1), (s), and (s_1) are the *only* separations of (b^1) which are *binomials*.

If now the last term of (s^1) be integrated by parts in the same manner as (2), we shall obtain (5) of the following summary, after solving for (b^1).

If the last term of (s) be integrated by parts similarly to (2), we shall obtain (4).

If the first term of (s) be integrated by parts similarly to (1), we shall obtain (3).

If the first term of (s_1) be integrated by parts similarly to (1), we shall obtain (6).

It is to be noticed that (3), (4), (5), and (6) are the *only* integrations by parts of (s^1), (s), and (s_1) which are binomials.

SUMMARY.

$$\int X^p x^{m-1} dx \quad \ldots \quad (b^1)$$

$$= \frac{X^p x^m}{m} - \frac{bnp}{m} \int X^{p-1} x^{m+n-1} dx \quad \ldots \quad (1)$$

$$= \frac{X^{p+1} x^{m-n}}{bn(p+1)} - \frac{m-n}{bn(p+1)} \int X^{p+1} x^{m-n-1} dx \quad \ldots \quad (2)$$

$$= \frac{X^{p+1} x^m}{am} - \frac{b(m+n+np)}{am} \int X^p x^{m+n-1} dx \quad \ldots \quad (3)$$

$$= \frac{-X^{p+1} x^n}{an(p+1)} + \frac{m+n+np}{an(p+1)} \int X^{p+1} x^{m-1} dx \quad \ldots \quad (4)$$

$$= \frac{X^p x^m}{m+np} + \frac{anp}{m+np} \int X^{p-1} x^{m-1} dx \quad \ldots \quad (5)$$

$$= \frac{X^{p+1} x^{m-n}}{b(m+np)} - \frac{a(m-n)}{b(m+np)} \int X^p x^{m-n-1} dx \quad \ldots \quad (6)$$

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